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ORIGINAL RESEARCH

Hyers–Ulam–Rassias stability of Cauchy–Jensen functional equation in RN-spaces

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Abstract Using the fixed point and direct methods, we prove the Hyers–Ulam stability of the following Cauchy–Jensen functional equation.

$$2f\left(\frac{\sum_{i=1}^{p} x_i + \sum_{j=1}^{q} y_j}{2} + \sum_{k=1}^{d} z_k\right) = \sum_{i=1}^{p} f(x_i) + \sum_{j=1}^{q} f(y_j) + 2\sum_{k=1}^{d} f(z_k)$$

where p, q, d are positive integers, in random normed spaces.

Keywords Hyers–Ulam–Rassias stability · Cauchy–Jensen functional equation · Fixed point method · Random normed spaces

Mathematics Subject Classification Primary 39B55

Introduction and preliminaries

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [32] in 1940. In the next year, Hyers [14] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [25] proved a generalization of Hyers's theorem for additive mappings. This new concept is known as generalized Hyers–Ulam stability or Hyers–Ulam–Rassias stability of functional equations. Furthermore, in 1994, a generalization of Rassias's theorem was obtained by Găvruta [13] by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$.

In 1983, a generalized Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [31] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group and, in 2002, Czerwik [9] proved the generalized Hyers–Ulam stability of the quadratic functional equation. The reader is referred to [1–29] and references therein for detailed information on stability of functional equations. In the sequel, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [30].

Throughout this paper, let Γ^+ denote the set of all probability distribution functions $F : R \cup [-\infty, +\infty] \to [0, 1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0, F(+\infty) = 1$. It is clear that the set $D^+ = \{F \in \Gamma^+ : l^- F(+\infty) = 1\}$, where $l^- f(x) = \lim_{t \to x^-} f(t)$, is a subset of Γ^+. The set Γ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in \mathbb{R}$. For any $a \ge 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t) = \begin{cases} 0 & \text{if} \quad t \le a, \\ 1 & \text{if} \quad t > a. \end{cases}$$

We can easily show that the maximal element in Γ^+ is the distribution function $H_0(t)$.

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Definition 1.1 A function $T:[0,1]^2 \to [0,1]$ is a *continuous triangular norm* (briefly, a t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(x,1)=x$ for all $x\in[0,1]$;
(d) $T(x,y)\le T(z,w)$ whenever $x\le z$ and $y\le w$ for all $x,y,z,w\in[0,1]$.

Three typical examples of continuous t-norms are as follows: $T(x,y)=xy, T(x,y)=\max\{a+b-1,0\}$, $T(x,y)=\min(a,b)$.

Recall that, if T is a t-norm and $\{x_n\}$ is a sequence in $[0,1]$, then $T^n_{i=1}x_i$ is defined recursively by $T^1_{i=1}x_1=x_1$ and $T^n_{i=1}x_i=T(T^{n-1}_{i=1}x_i,x_n)$ for all $n\ge 2$. $T^\infty_{i=n}x_i$ is defined by $T^\infty_{i=1}x_{n+i}$.

Definition 1.2 A *random normed space* (briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm and $\mu:X\to D^+$ is a mapping such that the following conditions hold:

(a) $\mu_x(t)=H_0(t)$ for all $x\in X$ and $t>0$ if and only if $x=0$;
(b) $\mu_{\alpha x}(t)=\mu_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha\in\mathbb{R}$ with $\alpha\neq 0$, $x\in X$ and $t\ge 0$;
(c) $\mu_{x+y}(t+s)\ge T(\mu_x(t),\mu_y(s))$ for all $x,y\in X$ and $t,s\ge 0$.

Definition 1.3 Let (X,μ,T) be an RN-space.

(a) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x\in X$ (write $x_n\to x$ as $n\to\infty$) if $\lim_{n\to\infty}\mu_{x_n-x}(t)=1$ for all $t>0$.
(b) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* in X if $\lim_{n\to\infty}\mu_{x_n-x_m}(t)=1$ for all $t>0$.
(c) The RN-space (X,μ,T) is said to be *complete* if every Cauchy sequence in X is convergent.

Theorem 1.4 [30] *If (X,μ,T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n\to x$, then* $\lim_{n\to\infty}\mu_{x_n}(t)=\mu_x(t)$.

Definition 1.5 Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a generalized metric on X if d satisfies the following conditions:

(a) $d(x,y)=0$ if and only if $x=y$ for all $x,y\in X$;
(b) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(c) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

Theorem 1.6 *Let (X,d) be a complete generalized metric space and $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then, for all $x\in X$, either*

$$d(J^nx,J^{n+1}x)=\infty \tag{1.1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(a) $d(J^nx,J^{n+1}x)<\infty$ *for all* $n_0\ge n_0$;
(b) *the sequence $\{J^nx\}$ converges to a fixed point y^* of J*;
(c) y^* is the unique fixed point of J in the set $Y=\{y\in X: d(J^{n_0}x,y)<\infty\}$;
(d) $d(y,y^*)\le\frac{1}{1-L}d(y,Jy)$ for all $y\in Y$.

In 1996, Hyers et al. [15] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [10, 22–24]).

This paper is organized as follows: In "Stability of the Cauchy–Jensen functional equation : a directmethod", using direct method, we prove the Hyers–Ulam–Rassias stability of the following functional equation that we call Cauchy–Jensen mapping

$$2f\left(\frac{\sum_{i=1}^p x_i+\sum_{j=1}^q y_j}{2}+\sum_{k=1}^d z_k\right)=\sum_{i=1}^p f(x_i)+\sum_{j=1}^q f(y_j)+2\sum_{k=1}^d f(z_k) \tag{1.2}$$

where $x_i,y_j,z_k\in X$, in random normed space. In "Stability of the Cauchy–Jensen functional equation : a fixed pointapproach", using the fixed point method, we prove the Hyers–Ulam–Rassias stability of the Cauchy–Jensen functional equation (1.2) in random normed spaces.

Stability of the Cauchy–Jensen functional equation: a direct method

For a given mapping $f:X\to Y$, we define

$$AQ_f(x,y,z)=2f\left(\frac{\sum_{i=1}^p x_i+\sum_{j=1}^q y_j}{2}+\sum_{k=1}^d z_k\right)-\sum_{i=1}^p f(x_i)-\sum_{j=1}^q f(y_j)-2\sum_{k=1}^d f(z_k)$$

for all $x_i,y_j,z_k\in X$.

In this section, using direct method, we prove the generalized Hyers–Ulam–Rassias stability of the Cauchy–Jensen additive functional equation (1.2) in random space .

Theorem 2.1 Let X be a real linear space $(Z,\mu',\min)$ be an RN-space and $\phi:X^{p+q+d}\to Z$ be a function such that there exists $0<\alpha<\frac{2}{p+q+2d}$ satisfying

$$\mu'_{\phi\left(\frac{2x_i}{p+q+2d},\frac{2y_j}{p+q+2d},\frac{2z_k}{p+q+2d}\right)}(t) \geq \mu'_{\alpha\phi(x_i,y_j,z_k)}(t) \tag{2.1}$$

for all $x_i, y_j, z_k \in X$ and $t > 0$ and

$$\lim_{n\to\infty} \mu'_{\phi\left(\frac{2^n x_i}{(p+q+2d)^n},\frac{2^n y_j}{(p+q+2d)^n},\frac{2^n z_k}{(p+q+2d)^n}\right)}\left(\frac{2^n t}{(p+q+2d)^n}\right) = 1$$

for all $x_i, y_j, z_k \in X$ and $t > 0$. Let $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying

$$\mu_{2f\left(\frac{\sum_{i=1}^p x_i+\sum_{j=1}^q y_j}{2}+\sum_{k=1}^d z_k\right)-\sum_{i=1}^p f(x_i)-\sum_{j=1}^q f(y_j)-2\sum_{k=1}^d f(z_k)}(t) \geq \mu'_{\phi(x_i,y_j,z_k)}(t) \tag{2.2}$$

for all $x_i, y_j, z_k \in X$ and $t > 0$, then the limit

$$L(x) = \lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n x}{(p+q+2d)^n}\right)$$

exists for all $x \in X$ and defines a unique Cauchy–Jensen mapping $L : X \to Y$ such that

$$\mu_{f(x)-L(x)}(t) \geq \mu'_{\phi(x,x,\ldots,x)}\left(\frac{(2-(p+q+2d)\alpha)t}{\alpha}\right) \tag{2.3}$$

for all $x \in X$ and $t > 0$.

Proof Putting $x_i = y_j = z_k = x$ in (2.2), we see that

$$\mu_{2f\left(\frac{p+q+2d}{2}x\right)-(p+q+2d)f(x)}(t) \geq \mu'_{\phi_{(x,x,\ldots,x)}}(t) \tag{2.4}$$

for all $x \in X$ and all $t > 0$. Replacing x by $\frac{2x}{p+q+2d}$ in (2.4), we obtain

$$\mu_{f(x)-\frac{p+q+2d}{2}f\left(\frac{2x}{p+q+2d}\right)}(t) \geq \mu'_{\phi\left(\frac{2x}{p+q+2d},\frac{2x}{p+q+2d},\ldots,\frac{2x}{p+q+2d}\right)}(2t) \tag{2.5}$$

for all $x \in X$ and all $t > 0$. Replacing x by $\frac{2^n x}{(p+q+2d)^n}$ in (2.5) and using (2.1), we obtain

$$\begin{aligned}
&\mu_{\frac{(p+q+2d)^{n+1}}{2^{n+1}}f\left(\frac{2^{n+1}x}{(p+q+2d)^{n+1}}\right)-\frac{(p+q+2d)^n}{2^n}f\left(\frac{2^n x}{(p+q+2d)^n}\right)}(t)\\
&\quad\geq \mu'_{\phi\left(\frac{2^{n+1}x}{(p+q+2d)^{n+1}},\frac{2^{n+1}x}{(p+q+2d)^{n+1}},\ldots,\frac{2^{n+1}x}{(p+q+2d)^{n+1}}\right)}\left(\frac{2^{n+1}t}{(p+q+2d)^n}\right)\\
&\quad\geq \mu'_{\phi(x,x,\ldots,x)}\left(\frac{2^{n+1}t}{(p+q+2d)^n\alpha^{n+1}}\right)
\end{aligned}$$

and so

$$\begin{aligned}
&\mu_{\frac{(p+q+2d)^n}{2^n}f\left(\frac{2^n x}{(p+q+2d)^n}\right)-f(x)}\left(\sum_{k=0}^{n-1}\frac{t(p+q+2d)^k\alpha^{k+1}}{2^{k+1}}\right)\\
&=\mu_{\sum_{k=0}^{n-1}2^{k+1}\frac{(p+q+2d)^{k+1}}{2^{k+1}}f\left(\frac{2^{k+1}x}{(p+q+2d)^{k+1}}\right)-\frac{(p+q+2d)^k}{2^k}f\left(\frac{2^k x}{(p+q+2d)^k}\right)}\\
&\quad\times\left(\sum_{k=0}^{n-1}\frac{t(p+q+2d)^k\alpha^{k+1}}{2^{k+1}}\right)\\
&\geq T_{k=0}^{n-1}\left(\mu'_{\frac{(p+q+2d)^{k+1}}{2^{k+1}}f\left(\frac{2^{k+1}x}{(p+q+2d)^{k+1}}\right)-\frac{(p+q+2d)^k}{2^k}f\left(\frac{2^k x}{(p+q+2d)^k}\right)}\left(\frac{t(p+q+2d)^k\alpha^{k+1}}{2^{k+1}}\right)\right)\\
&\geq T_{k=0}^{n-1}\left(\mu'_{\phi(x,x,\ldots,x)}(2t)\right) = \mu'_{\phi(x,x,\ldots,x)}(2t).
\end{aligned} \tag{2.6}$$

This implies that

$$\mu_{\frac{(p+q+2d)^n}{2^n}f\left(\frac{2^n x}{(p+q+2d)^n}\right)-f(x)}(t) \geq \mu'_{\phi(x,x,\ldots,x)}\left(\frac{2t}{\sum_{k=0}^{n-1}\frac{(p+q+2d)^k\alpha^{k+1}}{2^k}}\right). \tag{2.7}$$

Replacing x by $\frac{2^l x}{(p+q+2d)^l}$ in (2.7), we obtain

$$\begin{aligned}
&\mu_{\frac{(p+q+2d)^{n+l}}{2^{n+l}}f\left(\frac{2^{n+l}x}{(p+q+2d)^{n+l}}\right)-\frac{(p+q+2d)^l}{2^l}f\left(\frac{2^l x}{(p+q+2d)^l}\right)}(t)\\
&\geq \mu'_{\phi(x,x,\ldots,x)}\left(\frac{2t}{\sum_{k=l}^{n+l-1}\frac{(p+q+2d)^k\alpha^{k+1}}{2^k}}\right).
\end{aligned} \tag{2.8}$$

Since

$$\lim_{l,n\to\infty}\mu'_{\phi(x,x,\ldots,x)}\left(\frac{2t}{\sum_{k=l}^{n+l-1}\frac{(p+q+2d)^k\alpha^{k+1}}{2^k}}\right) = 1$$

it follows that $\left\{\frac{(p+q+2d)^n}{2^n}f\left(\frac{2^n x}{(p+q+2d)^n}\right)\right\}_{n\geq 1}$ is a Cauchy sequence in a complete RN-space $(Y, \mu, \min)$ and so there exists a point $L(x) \in Y$ such that

$$\lim_{n\to\infty}\frac{(p+q+2d)^n}{2^n}f\left(\frac{2^n x}{(p+q+2d)^n}\right) = L(x).$$

Fix $x \in X$ and put $p = 0$ in (2.8). Then we obtain

$$\mu_{\frac{(p+q+2d)^n}{2^n}f\left(\frac{2^n x}{(p+q+2d)^n}\right)-f(x)}(t) \geq \mu'_{\phi(x,x,\ldots,x)}\left(\frac{2t}{\sum_{k=0}^{n-1}\frac{(p+q+2d)^k\alpha^{k+1}}{2^k}}\right)$$

and so, for any $\epsilon > 0$,

$$\mu_{L(x)-f(x)}(t+\epsilon)\geq T\left(\mu_{L(x)-\frac{(p+q+2d)^n}{2^n}f\left(\frac{2^n x}{(p+q+2d)^n}\right)}(\epsilon),\mu_{\frac{(p+q+2d)^n}{2^n}f\left(\frac{2^n x}{(p+q+2d)^n}\right)-f(x)}(t)\right)$$
$$\geq T\left(\mu_{L(x)-\frac{(p+q+2d)^n}{2^n}f\left(\frac{2^n x}{(p+q+2d)^n}\right)}(\epsilon),\mu'_{\phi(x,x,\ldots,x)}\left(\frac{2t}{\sum_{k=0}^{n-1}\frac{(p+q+2d)^k\alpha^{k+1}}{2^k}}\right)\right). \tag{2.9}$$

Taking $n\to\infty$ in (2.9), we get

$$\mu_{L(x)-f(x)}(t+\epsilon)\geq\mu'_{\phi(x,x,\ldots,x)}\left(\frac{2t}{\sum_{k=0}^{\infty}\frac{(p+q+2d)^k\alpha^{k+1}}{2^k}}\right). \tag{2.10}$$

Since ϵ is arbitrary, by taking $\epsilon\to 0$ in (2.10), we get

$$\mu_{L(x)-f(x)}(t)\geq\mu'_{\phi(x,x,\ldots,x)}\left(\frac{(2-(p+q+2d)\alpha)t}{\alpha}\right).$$

Replacing x_i, y_j and z_k by $\frac{2^n x_i}{(p+q+2d)^n}$, $\frac{2^n y_j}{(p+q+2d)^n}$ and $\frac{2^n z_k}{(p+q+2d)^n}$ in (2.2), respectively, we get

$$\mu_{\frac{(p+q+2d)^n}{2^n}AQ_f\left(\frac{2^n x_i}{(p+q+2d)^n},\frac{2^n y_j}{(p+q+2d)^n},\frac{2^n z_k}{(p+q+2d)^n}\right)}(t)$$
$$\geq\mu'_{\phi\left(\frac{2^n x_i}{(p+q+2d)^n},\frac{2^n y_j}{(p+q+2d)^n},\frac{2^n z_k}{(p+q+2d)^n}\right)}\left(\frac{2^n t}{(p+q+2d)^n}\right)$$

for all $x_i, y_j, z_k\in X$ and $t>0$. Since

$$\lim_{n\to\infty}\mu'_{\phi\left(\frac{2^n x_i}{(p+q+2d)^n},\frac{2^n y_j}{(p+q+2d)^n},\frac{2^n z_k}{(p+q+2d)^n}\right)}\left(\frac{2^n t}{(p+q+2d)^n}\right)=1$$

we conclude that L satisfies (1.2).

To prove the uniqueness of the additive mapping L, assume that there exists another mapping $M:X\to Y$ which satisfies (2.3). Then we have

$$\mu_{L(x)-M(x)}(t)$$
$$=\lim_{n\to\infty}\mu_{\frac{(p+q+2d)^n}{2^n}L\left(\frac{2^n x}{(p+q+2d)^n}\right)-\frac{(p+q+2d)^n}{2^n}M\left(\frac{2^n x}{(p+q+2d)^n}\right)}(t)$$
$$\geq\lim_{n\to\infty}\min\left\{\mu_{\frac{(p+q+2d)^n}{2^n}L\left(\frac{2^n x}{(p+q+2d)^n}\right)-\frac{(p+q+2d)^n}{2^n}f\left(\frac{2^n x}{(p+q+2d)^n}\right)}\left(\frac{t}{2}\right),\right.$$
$$\left.\mu_{\frac{(p+q+2d)^n}{2^n}f\left(\frac{2^n x}{(p+q+2d)^n}\right)-\frac{(p+q+2d)^n}{2^n}L\left(\frac{2^n x}{(p+q+2d)^n}\right)}\left(\frac{t}{2}\right)\right\}$$
$$\geq\lim_{n\to\infty}\mu'_{\phi\left(\frac{2^n x}{(p+q+2d)^n},\frac{2^n x}{(p+q+2d)^n},\ldots,\frac{2^n x}{(p+q+2d)^n}\right)}$$
$$\times\left(\frac{2^n(2-(p+q+2d)\alpha)t}{2(p+q+2d)^n\alpha}\right)$$
$$\geq\lim_{n\to\infty}\mu'_{\phi(x,x,\ldots,x)}\left(\frac{2^n(2-(p+q+2d)\alpha)t}{2(p+q+2d)^n\alpha^{n+1}}\right).$$

Since

$$\lim_{n\to\infty}\frac{2^n(2-(p+q+2d)\alpha)t}{2(p+q+2d)^n\alpha^{n+1}}=\infty$$

we get

$$\lim_{n\to\infty}\mu'_{\phi(x,x,\ldots,x)}\left(\frac{2^n(2-(p+q+2d)\alpha)t}{2(p+q+2d)^n\alpha^{n+1}}\right)=1.$$

Therefore, it follows that $\mu_{L(x)-M(x)}(t)=1$ for all $t>0$ and so $L(x)=M(x)$. This completes the proof. $\square$

Corollary 2.2 Let X be a real normed linear space $(Z,\mu',\min)$ be an RN-space and $(Y,\mu,\min)$ be a complete RN-space. Let r is a positive real number with $0<r<1$, $z_0\in Z$ and $f:X\to Y$ be a mapping with $f(0)=0$ and satisfying

$$\mu_{2f\left(\frac{\sum_{i=1}^p x_i+\sum_{j=1}^q y_j}{2}+\sum_{k=1}^d z_k\right)-\sum_{i=1}^p f(x_i)-\sum_{j=1}^q f(y_j)-2\sum_{k=1}^d f(z_k)}(t)$$
$$\geq\mu'_{\left(\sum_{i=1}^p\|x_i\|^r+\sum_{j=1}^q\|y_j\|^r+\sum_{k=1}^d\|z_k\|^r\right)z_0}(t) \tag{2.11}$$

for all $x_i,y_j,z_k\in X$ and $t>0$. Then the limit $L(x)=\lim_{n\to\infty}\frac{(p+q+2d)^n}{2^n}f\left(\frac{2^n x}{(p+q+2d)^n}\right)$ exists for all $x\in X$ and defines a unique Cauchy–Jensen mapping $L:X\to Y$ such that

$$\mu_{f(x)-L(x)}(t)\geq\mu'_{\|x\|^r z_0}\left(\frac{(2(p+q+2d)^r-2^r(p+q+2d))t}{2^r(p+q+d)}\right)$$

for all $x\in X$ and $t>0$.

Proof Let $\alpha=\left(\frac{2}{p+q+2d}\right)^r$ and $\phi:X^{p+q+d}\to Z$ be a mapping defined by

$$\phi(x_i,y_j,z_k)=\left(\sum_{i=1}^p\|x_i\|^r+\sum_{j=1}^q\|y_j\|^r+\sum_{k=1}^d\|z_k\|^r\right)z_0.$$

Then, from Theorem 2.1, the conclusion follows. $\square$

Theorem 2.3 Let X be a real linear space $(Z,\mu',\min)$ be an RN-space and $\phi:X^{p+q+d}\to Z$ be a function such that there exists $0<\alpha<\frac{p+q+2d}{2}$ satisfying

$$\mu'_{\phi\left(\frac{(p+q+2d)x_i}{2},\frac{(p+q+2d)y_j}{2},\frac{(p+q+2d)z_k}{2}\right)}(t)\geq\mu'_{\alpha\phi(x_i,y_j,z_k)}(t) \tag{2.12}$$

for all $x_i, y_j, z_k\in X$ and $t>0$ and

$$\lim_{n\to\infty}\mu'_{\phi\left(\frac{(p+q+2d)^n x_i}{2^n},\frac{(p+q+2d)^n y_j}{2^n},\frac{(p+q+2d)^n z_k}{2^n}\right)}\left(\frac{(p+q+2d)^n t}{2^n}\right)=1$$

for all $x_i, y_j, z_k\in X$ and $t>0$. Let $(Y,\mu,\min)$ be a complete RN-space. If $f:X\to Y$ be a mapping with $f(0)=0$ and satisfying (2.2). Then the limit

$$L(x)=\lim_{n\to\infty}\frac{2^n}{(p+q+2d)^n}f\left(\frac{(p+q+2d)^n x}{2^n}\right)$$

exists for all $x\in X$ and defines a unique Cauchy–Jensen mapping $L:X\to Y$ such that

$$\mu_{f(x)-L(x)}(t)\geq\mu'_{\phi(x,x,\ldots,x)}(((p+q+2d)-2\alpha)t). \tag{2.13}$$

for all $x\in X$ and $t>0$.

Proof By (2.4), we have

$$\mu_{\frac{2}{p+q+2d}f\left(\frac{p+q+2d}{2}x\right)-f(x)}(t)\geq\mu'_{\phi_{(x,x,\ldots,x)}}((p+q+2d)t) \tag{2.14}$$

for all $x\in X$ and all $t>0$. Replacing x by $\frac{(p+q+2d)^n x}{2^n}$ in (2.14) and using (2.12), we obtain

$$\begin{aligned}&\mu_{\frac{2^{n+1}}{(p+q+2d)^{n+1}}f\left(\frac{(p+q+2d)^{n+1}x}{2^{n+1}}\right)-\frac{2^n}{(p+q+2d)^n}f\left(\frac{(p+q+2d)^n x}{2^n}\right)}(t)\\&\geq\mu'_{\phi\left(\frac{(p+q+2d)^n x}{2^n},\frac{(p+q+2d)^n x}{2^n},\ldots,\frac{(p+q+2d)^n x}{2^n}\right)}\left(\frac{(p+q+2d)^{n+1}t}{2^n}\right)\\&\geq\mu'_{\phi(x,x,\ldots,x)}\left(\frac{(p+q+2d)^{n+1}t}{2^n\alpha^n}\right).\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4 Let X be a real normed linear space $(Z,\mu',\min)$ be an RN-space and $(Y,\mu,\min)$ be a complete RN-space. Let r is a positive real number with $r>1$, $z_0\in Z$ and $f:X\to Y$ be a mapping with $f(0)=0$ and satisfying (2.11). Then the limit $L(x)=\lim_{n\to\infty}\frac{2^n}{(p+q+2d)^n}f\left(\frac{(p+q+2d)^n x}{2^n}\right)$ exists for all $x\in X$ and defines a unique Cauchy–Jensen mapping $L:X\to Y$ such that

$$\mu_{f(x)-L(x)}(t)\geq\mu'_{\|x\|^r z_0}\left(\frac{((p+q+2d)-2^r(p+q+2d)^{1-r})t}{(p+q+d)}\right)$$

for all $x\in X$ and $t>0$.

Proof Let $\alpha=\left(\frac{p+q+2d}{2}\right)^{1-r}$ and $\phi:X^{p+q+d}\to Z$ be a mapping defined by

$$\phi(x_i,y_j,z_k)=\left(\sum_{i=1}^{p}\|x_i\|^r+\sum_{j=1}^{q}\|y_j\|^r+\sum_{k=1}^{d}\|z_k\|^r\right)z_0.$$

Then, from Theorem 2.3, the conclusion follows. □

Stability of the Cauchy–Jensen functional equation: a fixed point approach

In the rest of the paper, by $\Phi(x_i,y_j,z_k)$, we mean that $\Phi(x_1,\ldots,x_p,y_1,\ldots,y_q,z_1,\ldots,z_d)$. Using the fixed point method, we prove the generalized Hyers–Ulam–Rassias stability of the functional equation $AQ_f(x,y,z)=0$ in random normed spaces.

Theorem 3.1 Let X be a linear space (Y,μ,T_M) be a complete RN-space and Φ be a mapping from X^{p+q+d} to $D^+(\Phi(x_i,y_j,z_k)$ is denoted by $\Phi_{x_i,y_j,z_k})$ such that there exists $0<\alpha<\frac{2}{p+q+2d}$ satisfying

$$\Phi_{\frac{2x_i}{p+q+2d},\frac{2y_j}{p+q+2d},\frac{2z_k}{p+q+2d}}(\alpha t)\geq\Phi_{x_i,y_j,z_k}(t) \tag{3.1}$$

for all $x_i,y_j,z_k\in X$ and $t>0$. Let $f:X\to Y$ be a mapping with $f(0)=0$ and satisfying

$$\begin{aligned}&\mu_{2f\left(\frac{\sum_{i=1}^{p}x_i+\sum_{j=1}^{q}y_j}{2}+\sum_{k=1}^{d}z_k\right)-\sum_{i=1}^{p}f(x_i)-\sum_{j=1}^{q}f(y_j)-2}\\&\quad\times\sum_{k=1}^{d}f(z_k)(t)\geq\Phi_{x_i,y_j,z_k}(t)\end{aligned} \tag{3.2}$$

for all $x_i,y_j,z_k\in X$ and $t>0$. Then the limit

$$\lim_{n\to\infty}\frac{(p+q+2d)^n}{2^n}f\left(\frac{2^n x}{(p+q+2d)^n}\right)=L(x)$$

exists for all $x\in X$ and $L:X\to Y$ is a unique mapping such that

$$\mu_{f(x)-L(x)}(t)\geq\Phi_{\underbrace{x,x,\ldots,x}_{(p+q+d)-\text{times}}}\left(\frac{2(1-\alpha)t}{\alpha}\right) \tag{3.3}$$

for all $x\in X$ and $t>0$.

Proof Putting $x_i=y_j=z_k=x$ in (2.2), we get

$$\mu_{2f\left(\frac{p+q+2d}{2}x\right)-(p+q+2d)f(x)}(t)\geq\Phi_{\underbrace{x,x,\ldots,x}_{(p+q+d)-\text{times}}}(t) \tag{3.4}$$

for all $x\in X$. Replacing x by $\frac{2x}{p+q+2d}$ in (3.4), we obtain

$$\begin{aligned}\mu_{f(x)-\frac{p+q+2d}{2}f\left(\frac{2x}{p+q+2d}\right)}(t)&\geq\Phi_{\underbrace{\frac{2x}{p+q+2d},\frac{2x}{p+q+2d},\ldots,\frac{2x}{p+q+2d}}_{(p+q+d)-\text{times}}}(2t)\\&\geq\Phi_{\underbrace{x,x,\ldots,x}_{(p+q+d)-\text{times}}}\left(\frac{2t}{\alpha}\right)\end{aligned} \tag{3.5}$$

for all $x\in X$ and all $t>0$.

Consider the set

$$S:=\{h:X\to Y;\ h(0)=0\}$$

and introduce the generalized metric on S:

$$d(g,h)=\inf_{u\in(0,+\infty)}\left\{\mu_{g(x)-h(x)}(ut)\geq\Phi_{\underbrace{x,x,\ldots,x}_{(p+q+d)-\text{times}}}(t),\ \forall x\in X\right\},$$

where, as usual, $\inf\phi=+\infty$. It is easy to show that (S,d) is complete (see [18]). Now we consider the linear mapping $J:(S,d)\to(S,d)$ such that

$$Jg(x) := \frac{p+q+2d}{2} g\left(\frac{2x}{p+q+2d}\right)$$

for all $x \in X$. First we prove that J is a strictly contractive mapping with the Lipschitz constant $\frac{p+q+2d}{2}\alpha$. In fact, Let $g, h \in S$ be given such that $d(g,h) < \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \Phi_{x,x,\ldots,x}(t)$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
&\mu_{Jg(x)-Jh(x)}\left(\frac{(p+q+2d)\alpha\varepsilon t}{2}\right) \\
&= \mu_{\frac{p+q+2d}{2}g\left(\frac{2x}{p+q+2d}\right)-\frac{p+q+2d}{2}h\left(\frac{2x}{p+q+2d}\right)}\left(\frac{(p+q+2d)\alpha\varepsilon t}{2}\right) \\
&= \mu_{g\left(\frac{2x}{p+q+2d}\right)-h\left(\frac{2x}{p+q+2d}\right)}(\alpha\varepsilon t) \\
&\geq \Phi_{\underbrace{\frac{2x}{p+q+2d},\ldots,\frac{2x}{p+q+2d}}_{(p+q+d)-\text{times}}}(\alpha t) \\
&\geq \Phi_{\underbrace{x,x,\ldots,x}_{(p+q+d)-\text{times}}}(t)
\end{aligned}$$

for all $x \in X$. So $d(g,h) < \varepsilon$ implies that $d(Jg, Jh) \leq \frac{(p+q+2d)\alpha\varepsilon}{2}$. This means that

$$d(Jg, Jh) \leq \frac{(p+q+2d)\alpha}{2} d(g,h)$$

for all $g, h \in S$.

It follows from (3.5) that $d(f, Jf) \leq \frac{\alpha}{2}$.

By Theorem 1.6, there exists a mapping $L : X \to Y$ satisfying the following:

(1) L is a fixed point of J, i.e.,

$$\frac{2L}{p+q+2d} = L\left(\frac{2x}{p+q+2d}\right) \tag{3.6}$$

for all $x \in X$. The mapping L is a unique fixed point of J in the set

$$M = \{g \in S : d(h,g) < \infty\}.$$

This implies that L is a unique mapping satisfying (3.6) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-h(x)}(ut) \geq \Phi_{\underbrace{x,x,\ldots,x}_{(p+q+d)-\text{times}}}(t)$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f, L) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n x}{(p+q+2d)^n}\right) = L(x)$$

for all $x \in X$;

(3) $d(f, L) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, L) \leq \frac{\alpha}{2(1-\alpha)}$$

and so

$$\mu_{f(x)-L(x)}\left(\frac{\alpha}{2(1-\alpha)}\right) \geq \Phi_{x,x,\ldots,x}(t)$$

for all $x \in X$ and all $t > 0$. This implies that the inequalities (3.3) hold. It follows from (3.1) and (3.2) that

$$\begin{aligned}
&\mu_{\frac{(p+q+2d)^n}{2^n}AQ_f\left(\frac{2^n x}{(p+q+2d)^n},\frac{2^n y}{(p+q+2d)^n},\frac{2^n z}{(p+q+2d)^n}\right)}(t) \\
&\geq \Phi_{\frac{2^n x_i}{(p+q+2d)^n},\frac{2^n y_j}{(p+q+2d)^n},\frac{2^n z_k}{(p+q+2d)^n}}\left(\frac{2^n t}{(p+q+2d)^n}\right) \\
&\geq \Phi_{x_i,y_j,z_k}\left(\frac{2^n t}{(p+q+2d)^n \alpha^n}\right)
\end{aligned}$$

for all $x_i, y_j, z_k \in X$ and all $t > 0$. Since

$$\lim_{n\to\infty} \Phi_{x_i,y_j,z_k}\left(\frac{2^n t}{(p+q+2d)^n \alpha^n}\right) = 1$$

for all $x_i, y_j, z_k \in X$ and all $t > 0$. So

$$\begin{aligned}
&\mu_{2L\left(\frac{\sum_{i=1}^p x_i + \sum_{j=1}^q y_j + 2\sum_{k=1}^d z_k}{2}\right) - \sum_{i=1}^p L(x_i)} - \sum_{j=1}^q L(y_j) \\
&- 2\sum_{k=1}^d L(z_k)(t) \\
&= 1
\end{aligned}$$

for all $x_i, y_j, z_k \in X$ and all $t > 0$. Hence $L : X \to Y$ is an Cauchy–Jensen mapping and we get desired results. □

Corollary 3.2 Let X be a real normed space, θ be a positive real number and r is a real number with $r > 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying

$$\begin{aligned}
&\mu_{2f\left(\frac{\sum_{i=1}^p x_i + \sum_{j=1}^q y_j}{2} + \sum_{k=1}^d z_k\right) - \sum_{i=1}^p f(x_i) - \sum_{j=1}^q f(y_j) - 2\sum_{k=1}^d f(z_k)}(t) \\
&\geq \frac{t}{t + \theta\left(\sum_{i=1}^p \|x_i\|^r + \sum_{j=1}^q \|y_j\|^r + \sum_{k=1}^d \|z_k\|^r\right)}
\end{aligned} \tag{3.7}$$

for all $x_i, y_j, z_k \in X$ and all $t > 0$. Then the limit $\lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n x}{(p+q+2d)^n}\right) = L(x)$ exists for all $x \in X$ and defines a unique Cauchy–Jensen mapping $L : X \to Y$ such that

$$\mu_{f(x)-L(x)}(t) \geq \frac{((p+q+2d)^r - 2^r)t}{((p+q+2d)^r - 2^r)t + 2^{r-1}(p+q+d)\theta\|x\|^r}$$

for all $x \in X$ and all $t > 0$.

Proof The proof follows from Theorem 2.1 by taking

$$\Phi_{x_i,y_j,z_k}(t) = \frac{t}{t+\theta\left(\sum_{i=1}^{p}\|x_i\|^r + \sum_{j=1}^{q}\|y_j\|^r + \sum_{k=1}^{d}\|z_k\|^r\right)}$$

for all $x_i, y_j, z_k \in X$ and all $t > 0$. Then we can choose $\alpha = \left(\frac{2}{p+q+2d}\right)^r$ and we get the desired result. □

Theorem 3.3 Let X be a linear space (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^{p+q+d} to D^+ ($\Phi(x_i, y_j, z_k)$ is denoted by Φ_{x_i,y_j,z_k}) such that there exists $0<\alpha<\frac{2}{p+q+2d}$ satisfying

$$\Phi_{\frac{2x_i}{p+q+2d},\frac{2y_j}{p+q+2d},\frac{2z_k}{p+q+2d}}(\alpha t) \geq \Phi_{x_i,y_j,z_k}(t)$$

for all $x_i, y_j, z_k \in X$ and all $t > 0$. Let $f: X \to Y$ be a mapping with $f(0) = 0$ and satisfying (3.2). Then the limit

$$\lim_{n\to\infty}\frac{2^n}{(p+q+2d)^n} f\left(\frac{(p+q+2d)^n x}{2^n}\right) = L(x)$$

exists for all $x \in X$ and $L: X \to Y$ is a unique Cauchy–Jensen additive mapping such that

$$\mu_{f(x)-L(x)}(t) \geq \Phi_{\underbrace{x,x,\ldots,x}_{(p+q+d)-\text{times}}}((p+q+2d)(1-\alpha)t) \tag{3.8}$$

for all $x \in X$.

Proof Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J: (S, d) \to (S, d)$ such that

$$Jg(x) := \frac{2}{p+q+2d} g\left(\frac{p+q+2d}{2}x\right)$$

for all $x \in X$.

It follows from (3.4) that $d(f, Jf) \leq \frac{1}{p+q+2d}$. By Theorem 1.6, there exists a mapping $L: X \to Y$ satisfying the following:

(1) L is a fixed point of J, i.e.,

$$\frac{(p+q+2d)L}{2} = L\left(\frac{(p+q+2d)x}{2}\right) \tag{3.9}$$

for all $x \in X$. The mapping L is a unique fixed point of J in the set

$$M = \{g \in S : d(h, g)<\infty\}.$$

This implies that L is a unique mapping satisfying (3.9) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-h(x)}(ut) \geq \Phi_{x,x,\ldots,x}(t)$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f, L) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty}\frac{2^n}{(p+q+2d)^n} f\left(\frac{(p+q+2d)^n x}{2^n}\right) = L(x)$$

for all $x \in X$;

(3) $d(f, L) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, L) \leq \frac{1}{(p+q+2d)(1-\alpha)}$$

and so

$$\mu_{f(x)-L(x)}\left(\frac{t}{(p+q+2d)(1-\alpha)}\right) \geq \Phi_{\underbrace{x,x,\ldots,x}_{(p+q+d)-\text{times}}}(t)$$

for all $x \in X$ and all $t > 0$. This implies that the inequalities (3.8) hold. The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.4 Let X be a real normed space, θ be a positive real number and r be a real number with $0<r<1$. Let $f: X \to Y$ be a mapping with $f(0) = 0$ and satisfying (3.7). Then there exists a unique Cauchy–Jensen mapping $L: X \to Y$ such that

$$\mu_{f(x)-L(x)}(t) \geq \frac{(p+q+2d)(2^r-(p+q+2d)^r)t}{(p+q+2d)(2^r-(p+q+2d)^r)t + 2^r(p+q+d)\theta\|x\|^r}$$

for all $x \in X$ and all $t > 0$.

Proof The proof follows from Theorem 3.3 by taking

$$\Phi_{x_i,y_j,z_k}(t) = \frac{t}{t+\theta\left(\sum_{i=1}^{p}\|x_i\|^r + \sum_{j=1}^{q}\|y_j\|^r + \sum_{k=1}^{d}\|z_k\|^r\right)}$$

for all $x_i, y_j, z_k \in X$ and all $t > 0$. Then we can choose $\alpha = \left(\frac{p+q+2d}{2}\right)^r$ and we get the desired result. □

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